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The Traveling Salesman Problem and Its Analogy with Two-Dimensional Spin Glasses

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The low-temperature regime of the Euclidean traveling salesman problem is studied numerically. The specific heat behavior is analyzed in terms of the number of cities and compared with that of spin-glasses. A properly defined order parameter shows the existence of freezing effects at low temperature. The Euclidean TSP behaves as a spin glass in two dimensions.

KEY WORDS: Optimization problem; two-dimensional spin glasses.

The traveling salesman problem (TSP) has been studied for a long time as the archetype of the nondeterministic polynomial time (NP-complete) class of problems. Because of the strategic importance of the TSP, great effort has been made to find approximate solutions. Analytical and numerical techniques developed in the field of spin glasses have been applied to it succesfully because of the similarity between the low-temperature behavior of spin glasses and several complex combinatorial optimization problems.[1]

The TSP attempts to find the shortest route that connects all N given points (or cities), the route ending at its starting point. Two different versions of this problem have been studied. One is the nonmetric TSP, in which the distances between the cities are considered as independent random variables with a certain probability distribution. This approach is particulary suitable for analytical calculations.[2,3] It can be considered as the mean-field version, since the dimension does not play any role. The numerical work of Toulouse and Kirkpatrick[4] for this model with a uniform distribution suggests that there is no phase transition at finite tem-

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perature, though they found abundant evidence of freezing. It is known, however, that there is a phase transition at $T \neq 0$ for a suitable distribution of distances.[2] This means that the probability distribution is relevant to the behavior of the model.

The other version is the Euclidean one, in which the cities form a collection of points on a plane. The distance $d(i, j)$ between the pair (i, j) is the usual Euclidean one. The statistical mechanics of this version has been less studied,[5,6] leaving open the question of whether there might be a phase transition at some finite temperature. In this paper we analyze numerically the low-temperature behavior of the Euclidean TSP and compare these results with those of spin glasses.

The cost function is the length of the tour

$$l = \sum_{i=1}^{N} d(p(i), p(i+1))$$

where $p(i)$ is some permutation of the N cities and $\{d(i, j)\}$ is the distance matrix of the cities. The Monte Carlo algorithm can be easily generalized to optimization problems, simply by replacing the energy in the transition probability by the cost function. An important point of the method is the choice of its local rearrangements or local moves. We considered the strategy of 2-opt replacement, which consists in exchanging the position in the order of the tour of two randomly chosen cities. One MC step is defined as N attempted random rearrangements.

The energy was computed for $N = 49$, 81, 100, 144, 196, and 400. For each value of N we considered three different instances of the problem.

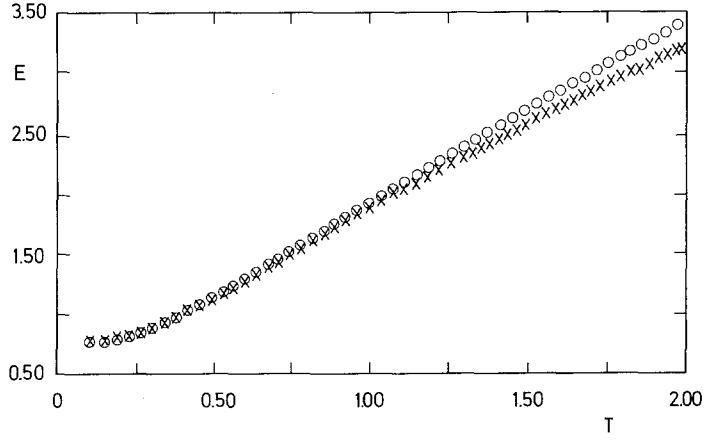

Fig. 1. Energy as a function of T for $N = (\times)$ 196, $(\bigcirc)$ 400.

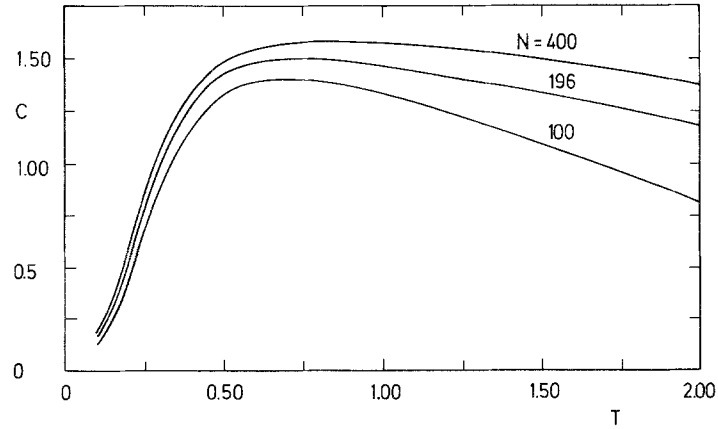

Fig. 2. Specific heat C as a function of T for $N = 100, 196, 400$.

Starting from a random configuration, the first run was taken at $T = 20.0$ in scaled units ($T = t\sqrt{N}$). Also, the length was scaled as $L = 1/\sqrt{N}$. The system was then cooled in steps ΔT which take the values 0.1 away from $T = 0$ and 0.01 near $T = 0$. This means that the cooling rate Q used, defined as the time in MC steps (m) spent at each temperature, $Q = \Delta T/m$, depends on the temperature range. For $T > 2.0$ the cooling rate was chosen to be 10^{-4}; for a lower temperature it was 10^{-5}; and in the range $0.01 < T < 0.1$ we considered $Q = 10^{-6}$. We measured the energy explicitly with three different temporal MC intervals to check that the energy measurements were not correlated. The number of MC steps needed to reach equilibrium was determined empirically so that the energy has clearly stopped drifting. The results of the energy for different sample sizes are shown in Fig. 1.

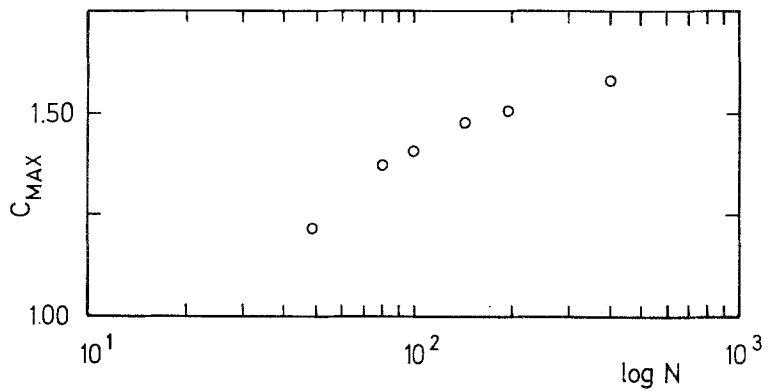

Fig. 3. The maximum of C as a function of $\log N$.

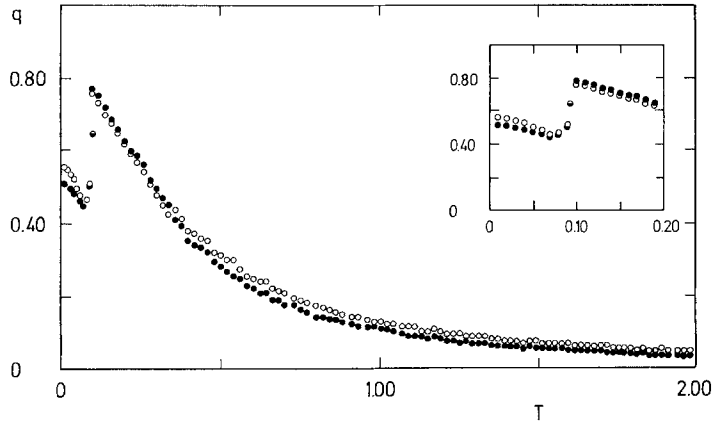

Fig. 4. The order parameter q as a function of T for two values of $N = (\bigcirc)\,100$ and $(\bullet)\,196$. The cooling rate for $T > 0.1$ was $Q = 10^{-5}$, and for $T < 0.1$ was $Q = 10^{-6}$. In the upper corner the zone of interest is shown in detail.

There are two ways of computing the specific heat C from an MC simulation. The first is to calculate it from the fluctuations of the length, or energy, L, so $C \sim \langle L^2 \rangle - \langle L \rangle^2$. This method was tried, but the results were strongly dependent on the way the computation was done. This is because this method involves subtracting two large, fluctuating quantities. The second way to compute C is by differentiation of the energy with respect to the temperature. We fited the low-energy curve with a least squares method and evaluated the derivative. The results for the specific heat C are shown in Fig. 2. We found a rounded peak similar to the two-dimensional E-A spin-glass model.[7,8] By studying also the dependence of the maximum of C on N (see Fig. 3), we found its behavior to be qualitatively similar to that of the E-A spin glass in 2D (see Fig. 18 in ref. 8).

Following Morgenstern and Wurtz,[9] we define an order parameter q as the number of bonds that do not change between the initial configuration (after equilibrium was reached) and the final one at a given temperature. Although the behavior of the peak of C with N might suggest a phase transition at a finite temperature, we note, observing the order parameter q, that it is a freezing effect, similar to that in the spin-glass case. In fact, changing the observation time, with the same T, near $T = 0$, we observe a drastic change in the behavior of q (see Fig. 4). It can be expected that if the cooling procedure were arbitrarily slow, then the order parameter q would be equal to zero for all finite temperatures.

It must be emphasized that the observed fluctuations from sample to sample were negligible for all the studied quantities and sizes.

In conclusion, our numerical data suggest that the Euclidean TSP

behaves qualitatively like a spin glass in 2D. If this is so, the results on the phase transition in the last model would be also valid for the Euclidean TSP. One should remark that the E-A spin glass in 2D is a P-problem,[10] while the TSP is an NP-problem, which in fact seems to have no relevance to the thermodynamic properties of the models.

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